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Multivariate Partial Newton–Padé and Newton–Padé Type Approximants

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The notion of partial Padé approximant is generalized to that of general order multivariate partial Newton–Padé approximant. Previously introduced multivariate Padé-type approximants are recaptured as special cases so that it is a true and unifying generalization. The last section contains numerical results for the bivariate Beta function. © 1993 Academic Press, Inc.

1. THE MULTIVARIATE NEWTON–PADÉ APPROXIMATION PROBLEM

We shall often restrict our description to the bivariate case for the sake of notational simplicity although we use the term multivariate. Let a bivariate function $f(x, y)$ be known in the points $(x_i, y_j) \in \mathbb{C}^2$ with $(i, j) \in I$, a finite subset of $\mathbb{N}^2$ playing the role of index set. If none of the points in $\{(x_i, y_j)\}_{(i,j)\in I}$ coincide then we are dealing with a rational interpolation problem and the values in $\{f_{ij}\}_{(i,j)\in I}$ are function values. If all the interpolation points coincide then the problem is one of Padé approximation and it is well known that the given data are not function values but Taylor coefficients. If some of the points coincide and some do not then the problem is of a mixed type and it is called a rational Hermite interpolation problem or a Newton–Padé approximation problem. In [8] is indicated how one should interpret the data f_{ij}: some of them are partial derivatives and some of them are function values. In the sequel of the text we shall

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distinguish, when necessary, between the Padé approximation case where all the interpolation points coincide and the Newton–Padé approximation case where this is not so.

With our data points (x_i, y_j) we construct the polynomial basis functions

$$B_{ij}(x, y) = \prod_{k=0}^{i-1} (x - x_k) \sum_{l=0}^{j-1} (y - y_l).$$

The problem of interpolating the data f_{ij} by a bivariate rational function was formulated in [8] as follows. Choose finite subsets N (from "Numerator") and D (from "Denominator") of $\mathbb{N}^2$ with $N \subset I$ and compute bivariate polynomials

$$\begin{aligned} p(x, y) &= \sum_{(i,j)\in N} a_{ij} B_{ij}(x, y), \qquad \#N = n+1 \\ q(x, y) &= \sum_{(i,j)\in D} b_{ij} B_{ij}(x, y), \qquad \#D = m+1 \end{aligned} \tag{1a}$$

such that

$$(fq - p)(x_i, y_j) = 0, \qquad (i, j) \in I, \qquad \#I = n + m + 1. \tag{1b}$$

If $q(x_i, y_j) \neq 0$ then this last condition implies that

$$f(x_i, y_j) = \frac{p}{q}(x_i, y_j), \qquad (i, j) \in I.$$

If some of the x_i and y_j coincide then also higher partial derivatives of $(fq - p)$ will cancel at (x_i, y_j) and higher partial derivatives of f will agree with those of p/q at (x_i, y_j) [8]. The following two conditions for the polynomials given in (1a) are sufficient to satisfy (1b), both in the Padé and the Newton–Padé approximation case [8],

$$(fq - p)(x, y) = \sum_{(i,j)\in \mathbb{N}^2 \setminus I} d_{ij} B_{ij}(x, y) \tag{2a}$$

$$I \text{ satisfies the inclusion property,} \tag{2b}$$

where the series development (2a) is still formal and where (2b) means that when a point (i, j) belongs to I, all the points in the rectangle emanating from the origin with (i, j) as its furthermost corner belong to I. How this can be achieved in a lot of situations is explained in [8]. From now on we denote a rational function $(p/q)(x, y)$ satisfying (1) or (2) for data coming from the function $f(x, y)$ by $[N/D]_I^f$.

By the set $N * D$ we denote the index set that results from the multiplication of a polynomial indexed by N with a polynomial indexed by D. Since we work with the polynomial basis functions $B_{ij}(x, y)$ instead of $x^i y^j$, we must keep in mind that

$$B_{ij}(x, y)\, B_{kl}(x, y) = \sum_{\mu=0}^{k} \sum_{\nu=0}^{l} \lambda_{\mu\nu} B_{i+\mu, j+\nu}(x, y).$$

So

$$N * D = \bigcup_{(i,j)\in N} \bigcup_{(k,l)\in D} ([i, i+k] \times [j, j+l] \cap \mathbb{N}^2)$$
$$\supset \{(i+k, j+l) \mid (i, j)\in N, (k, l)\in D\}.$$

In case all the interpolation points (x_i, y_j) coincide, for example, and for simplicity in $(0, 0)$, then

$$N * D = \{(i+k, j+l) \mid (i, j)\in N, (k, l)\in D\}.$$

If moreover the sets N and D satisfy the inclusion property, then

$$N \cup D \subset N * D.$$

2. General Order Multivariate Partial Newton–Padé Approximants

The notion of partial Padé approximant was introduced by Brezinski [3] for univariate functions $f(x)$: some of the Padé approximation conditions are dropped due to the knowledge of some poles or zeros of $f(x)$. Let the polynomials $v_k(x)$ and $w_l(x)$ respectively represent k zeros and l poles of f. The partial Padé approximation problem for f consists in finding polynomials $p(x)$ and $q(x)$, respectively, of degree n and m and satisfying

$$(fqw_l - pv_k)(x) = O(x^{n+m+1}). \tag{3}$$

The rational function $(pv_k)/(qw_l)$ is then called the partial Padé approximant to f of order $(n+k, m+l)$. It is easy to see that, if $v_k(0) \neq 0$, the rational function p/q is the Padé approximant of order (n, m) to fw_l/v_k [3]. We generalize this concept as follows.

Let the polynomials $V_k(x, y)$ and $W_l(x, y)$ respectively represent some knowledge about the zeros and poles of $f(x, y)$,

$$V_k(x, y) = \sum_{(i,j)\in V} v_{ij} B_{ij}(x, y)$$
$$\# V = k + 1$$

$$W_l(x, y) = \sum_{(i,j) \in W} w_{ij} B_{ij}(x, y)$$
$$\# W = l + 1.$$

Consider the following approximation problem. Let the finite subset $I \subset \mathbb{N}^2$ index those data points (x_i, y_j) that will be used as interpolation points. The knowledge of (x, y) in these interpolation points (x_i, y_j) can be expressed by means of a formal Newton series development for f,

$$f(x, y) = \sum_{(i,j) \in \mathbb{N}^2} c_{ij} B_{ij}(x, y)$$
$$c_{ij} = f[x_0, ..., x_i][y_0, ..., y_j],$$

where the bivariate divided differences with possible coalescence of coordinates are computed as in [8]. To generalize (3) we look for polynomials

$$P(x, y) = \sum_{(i,j) \in N} a_{ij} B_{ij}(x, y) \qquad N \text{ from "Numerator"}$$

$$Q(x, y) = \sum_{(i,j) \in D} b_{ij} B_{ij}(x, y) \qquad D \text{ from "Denominator"}$$

satisfying

$$(fQW_l - PV_k)(x, y) = \sum_{(i,j) \in \mathbb{N}^2 \setminus I} d_{ij} B_{ij}(x, y). \tag{4}$$

Which conditions have to be imposed on N and D to find a nontrivial solution for the unknowns a_{ij} and b_{ij} in (4)? Assuming that $V_k(x_i, y_j) \neq 0$ we first study

$$\left(\frac{fW_l}{V_k} Q - P\right)(x, y) = \sum_{(i,j) \in \mathbb{N}^2 \setminus I} e_{ij} B_{ij}(x, y),$$

where

$$\left(\frac{fW_l}{V_k}\right)(x, y) = \sum_{(i,j) \in \mathbb{N}^2} \tilde{c}_{ij} B_{ij}(x, y). \tag{5}$$

The coefficients $\tilde{c}_{ij}$ are given by

$$\gamma_{ij} = (fW_l)[x_0, ..., x_i][y_0, ..., y_j]$$
$$= \sum_{(t,u) \in W} W_l[x_0, ..., x_t][y_0, ..., y_u] f[x_t, ..., x_i][y_u, ..., y_j]$$
$$= \sum_{(t,u) \in W} w_{tu} f[x_t, ..., x_i][y_u, ..., y_j]$$

and

$$\gamma_{ij} = \left[\left(\frac{fW_l}{V_k}\right) V_k\right][x_0, ..., x_i][y_0, ..., y_j]$$

$$= \sum_{\mu=0}^{i} \sum_{\nu=0}^{j} \tilde{c}_{\mu\nu} V_k[x_\mu, ..., x_i][y_\nu, ..., y_j].$$

If P/Q is a Newton–Padé approximant to $(fW_l/V_k)(x, y)$ then $(PV_k)/(QW_l)$ is a partial Newton–Padé approximant to $f(x, y)$. Here a Newton–Padé approximant P/Q is denoted by $[N/D]_I^{fW_l/V_k}$ and we introduce the notation $\{N, V/D, W\}_I^f$ for a partial Newton–Padé approximant $(PV_k)/(QW_l)$. From [8] we know that $[N/D]_I^{fW_l/V_k}$ can be computed for

$$N \subset I$$

$$\#(I \backslash N) = \#D - 1$$

I inclusion property.

Using known results for general order multivariate Newton–Padé approximants, a partial Newton–Padé approximant $\{N, V/D, W\}_I^f$ can be expressed as a ratio of determinants involving the coefficients $\tilde{c}_{ij}$ from the formal Newton series expansion (5) for $(fW_l/V_k)(x, y)$. Let us number the indices in D by (d_0, e_0), (d_1, e_1), ..., (d_m, e_m) and the indices in $I \backslash N$ by (h_1, k_1), ..., (h_m, k_m). If the rank of the coefficient matrix of the linear conditions arising from (2a) [8] with f replaced by (fW_l/V_k), is maximal, then $Q(x, y)$ and $P(x, y)$ are given by

$$Q(x, y) = \begin{vmatrix} B_{d_0 e_0}(x, y) & \cdots & B_{d_m e_m}(x, y) \\ \tilde{c}_{d_0 h_1, e_0 k_1} & \cdots & \tilde{c}_{d_m h_1, e_m k_1} \\ \vdots & & \vdots \\ \tilde{c}_{d_0 h_m, e_0 k_m} & \cdots & \tilde{c}_{d_m h_m, e_m k_m} \end{vmatrix} \tag{6a}$$

$$P(x, y) = \begin{vmatrix} \sum_{(i,j) \in N} \tilde{c}_{d_0 i, e_0 j} B_{ij}(x, y) & \cdots & \sum_{(i,j)} \tilde{c}_{d_m i, e_m j} B_{ij}(x, y) \\ \tilde{c}_{d_0 h_1, e_0 k_1} & \cdots & \tilde{c}_{d_m h_1, e_m k_1} \\ \vdots & & \vdots \\ \tilde{c}_{d_0 h_m, e_0 k_m} & \cdots & \tilde{c}_{d_m h_m, e_m k_m} \end{vmatrix}. \tag{6b}$$

The error formulas developed in [1] for general order multivariate Newton–Padé approximants remain valid when applied to the function fW_l/V_k. When calculating a partial Padé approximant instead of a partial Newton–Padé approximant, we use a formal Taylor series development of

$(fW_l/V_k)(x, y)$ and carry out the same computations as above. In this case all the interpolation points coincide in one point. For special results about general order partial Padé approximants we refer to the next section.

3. Algebraic Properties of the Multivariate Partial Padé Approximant

It is well known that univariate Padé approximants satisfy a number of covariance properties, such as reciprocal covariance, homographic covariance, covariance for some transformations of the variable. These covariance properties remain valid for univariate partial Padé approximants, as was pointed out by Brezinski in [3]. In this section we study the covariance properties of the multivariate partial Padé approximant. For the sake of notational simplicity we stick to the bivariate case. Let the formal Taylor series development of $f(x, y)$ be given by

$$f(x, y) = \sum_{(i,j) \in \mathbb{N}^2} c_{ij} x^i y^j$$

with

$$c_{00} \neq 0.$$

Then the formal Taylor series development of $g(x, y) = (1/f)(x, y)$ is defined by

$$g(x, y) = \sum_{(i,j) \in \mathbb{N}^2} d_{ij} x^i y^j$$

with

$$f(x, y)\, g(x, y) = 1.$$

If the polynomial $V_k(x, y)$ contains information on the zeros of f, then it also contains information on the poles of g and vice versa for $W_l(x, y)$. If

$$(fQW_l - PV_k)(x, y) = \sum_{(i,j) \in \mathbb{N}^2 \setminus I} d_{ij} x^i y^j$$

then after multiplication by $-g(x, y)$, we obtain

$$(gPV_k - QW_l)(x, y) = \sum_{(i,j) \in \mathbb{N}^2 \setminus I} \tilde{d}_{ij} x^i y^j.$$

From this we can conclude

THEOREM 1. *Let* $\{N, V/D, W\}_I^f$ *be a general order multivariate partial Padé approximant to* $f(x, y)$ *as defined above and let* $g(x, y) = (1/f)(x, y)$. *Then*

$$\{N, V/D, W\}_I^f \cdot \{D, W/N, V\}_I^g = 1.$$

By multiplying the formal Taylor series expansion of $f(x, y)$ by a constant complex number, we do not change its zeros or poles. It is easy to verify

THEOREM 2. *Let* $\{N, V/D, W\}_I^f$ *be a general order multivariate partial Padé approximant to* $f(x, y)$ *as defined above and let* $a \neq 0$. *Then*

$$\{N, V/D, W\}_I^{af} = a\{N, V/D, W\}_I^f.$$

If we study the homographic function covariance of the multivariate partial Padé approximant, we must disappoint the reader. By transforming the function f into the function $\tilde{f} = (af + b)/(cf + d)$, the rational approximant under consideration transforms into

$$\frac{aPV_k + bQW_l}{cPV_k + dQW_l}(x, y) = \frac{\sum_{(ij) \in \tilde{N}} \tilde{a}_{ij} x^i y^j}{\sum_{(ij) \in \tilde{D}} \tilde{b}_{ij} x^i y^j}$$

which cannot necessarily be written in the form $(\tilde{P}V_k)/(\tilde{Q}W_l)$ with $\tilde{P}/\tilde{Q} = [N/D]_I^{\tilde{f}W_l/V_k}$.

Let us now study some changes in the variables. We define

$$a \neq 0, \qquad b \neq 0$$
$$\tilde{V}_k(x, y) = V_k(ax, by)$$
$$\tilde{W}_l(x, y) = W_l(ax, by)$$
$$\tilde{f}(x, y) = f(ax, by).$$

Since

$$(fQW_l - PV_k)(x, y) = \sum_{(i,j) \in \mathbb{N}^2 \setminus I} d_{ij} x^i y^j$$

implies

$$(fQW_l - PV_k)(ax, by) = \sum_{(i,j) \in \mathbb{N}^2 \setminus I} d'_{ij} x^i y^j$$

it is easy to conclude

THEOREM 3. *Let* $\{N, V/D, W\}_I^f$ *be a general order multivariate partial Padé approximant to* $f(x, y)$ *as defined above and let* $a \neq 0$, $b \neq 0$ *with* $\tilde{f}$, $\tilde{W}_l$ *and* $\tilde{V}_k$ *as defined above. Then*

$$\{N, V/D, W\}_I^{\tilde{f}} = \{N, V/D, W\}_I^f(ax, by).$$

Another change is the translation of the coefficients in the formal power series, in other words a multiplication of $f(x, y)$ by $x^s y^t$. We introduce the notations

$$\hat{N} = N + \{(s, t)\} = \{(i+s, j+t) \mid (i, j) \in N\}$$

$$\hat{I} = \overline{I + \{(s, t)\}} = \bigcup_{(k,l) \in I} ([0, k+s] \times [0, l+t] \cap \mathbb{N}^2).$$

Here overlining means making the set satisfy the inclusion property, in other words taking a kind of closure, namely filling the "holes" when looking at the set in $\mathbb{N}^2$. If $N \subset I$ then also $\hat{N} \subset \hat{I}$. One can verify

THEOREM 4. *Let* $\{N, V/D, W\}_I^f$ *be a general order multivariate partial Padé approximant to* $f(x, y)$ *as defined above and let* $\hat{N}$ *and* $\hat{I}$ *be defined as above. Then*

$$\{\hat{N}, V/D, W\}_{\hat{I}}^{x^s y^t f} = x^s y^t \{N, V/D, W\}_I^f.$$

The most important change of variable is the one involved in the homographic variable covariance of the multivariate partial Padé approximant.

THEOREM 5. *Let* $\{N, V/D, W\}_I^f$ *be a general order multivariate partial Padé approximant to* $f(x, y)$ *as defined above with* $N = D = ([0, i_M] \times [0, j_M]) \cap \mathbb{N}^2$ *and let*

$$\tilde{x} = \frac{ax + b}{cx + d}, \qquad \tilde{y} = \frac{a'y + b'}{c'y + d'}$$

$$\tilde{V}_k(x, y) = (cx + d)^{k_M} (c'y + d')^{l_M} V_k(\tilde{x}, \tilde{y})$$

$$\tilde{W}_l(x, y) = (cx + d)^{k_M} (c'y + d')^{l_M} W_l(\tilde{x}, \tilde{y})$$

$$\tilde{f}(x, y) = f(\tilde{x}, \tilde{y})$$

$$ad - bc \neq 0, \qquad a'd' - b'c' \neq 0$$

with $V = W = ([0, k_M] \times [0, l_M]) \cap \mathbb{N}^2$. *Then*

$$\{N, V/D, W\}_I^{\tilde{f}}(x, y) = \{N, V/D, W\}_I^f(\tilde{x}, \tilde{y}).$$

Proof. For $ad-bc \neq 0$,

$$\begin{aligned}\tilde{x} &= \frac{1}{d}(ax+b)\sum_{i=0}^{\infty}(-1)^i\left(\frac{c}{d}x\right)^i \\ &= \frac{b}{d}+x\left(\frac{a}{d}-\frac{bc}{d^2}\right)+\cdots \\ &= \frac{b}{d}+\frac{x}{d^2}(ad-bc)+\cdots\end{aligned}$$

represents a formal power series in x and analogously for $\tilde{y}$. Equation (4) combined with these formal power series expansions for $\tilde{x}$ and $\tilde{y}$ results in

$$\begin{aligned}&(\tilde{f}\tilde{W}_l)(x, y)\,Q(\tilde{x}, \tilde{y}) - \tilde{V}_k(x, y)\,P(\tilde{x}, \tilde{y}) \\ &\quad = (cx+d)^{kM}(c'y+d')^{lM}\sum_{(i,j)\in\mathbb{N}^2\setminus I} d_{ij}\tilde{x}^i\tilde{y}^j \\ &\quad = \sum_{(i,j)\in\mathbb{N}^2\setminus I}\tilde{d}_{ij}x^iy^j.\end{aligned}$$

The fact that

$$\begin{aligned}\frac{P(\tilde{x}, \tilde{y})}{Q(\tilde{x}, \tilde{y})} &= \frac{\sum_{(i,j)\in N} a_{ij}\tilde{x}^i\tilde{y}^j}{\sum_{(i,j)\in N} b_{ij}\tilde{x}^i\tilde{y}^j} \\ &= \frac{(cx+d)^{iM}(c'y+d')^{jM}\sum_{(i,j)\in M} a_{ij}\tilde{x}^i\tilde{y}^j}{(cx+d)^{iM}(c'y+d')^{jM}\sum_{(i,j)\in M} b_{ij}\tilde{x}^i\tilde{y}^j} \\ &= \frac{\sum_{(r,s)\in N}\tilde{a}_{rs}x^ry^s}{\sum_{(r,s)\in N}\tilde{b}_{rs}x^ry^s}\end{aligned}$$

completes the proof. ∎

Last but not least we have the consistency property. If we are given an irreducible rational function $f(x, y)$ right from the start, do we come across it when calculating the appropriate general order Newton–Padé approximant. By this we mean that for

$$\begin{aligned}f(x, y) = \frac{g(x, y)}{h(x, y)} &= \frac{\sum_{(i,j)\in N} g_{ij}x^iy^j}{\sum_{(i,j)\in D} h_{ij}x^iy^j} \\ V_k(x, y) = 1, &\qquad W_l(x, y) = 1 \\ (fQ-P)(x, y) &= \sum_{(i,j)\in\mathbb{N}^2\setminus I} d_{ij}B_{ij}(x, y)\end{aligned}$$

with $P(x, y)$ and $Q(x, y)$ defined by (1a), we want to find

$$(Ph - gQ)(x, y) = 0.$$

It is clear that this is the case if the general order multivariate Newton–Padé approximation problem $[N/D]_I^f$ has a unique solution, because then both P/Q and g/h satisfy the approximation conditions (4). If the solution is nonunique we can get in trouble because of the nonunicity of the irreducible form of the Newton–Padé approximant. A solution of the form

$$\frac{\alpha + \alpha x + (1 - \alpha) y}{1 + x + y}$$

has three different irreducible forms, namely

$$\alpha = 0.0: \quad \frac{y}{1 + x + y}$$

$$\alpha = 0.5: \quad 0.5$$

$$\alpha = 1.0: \quad \frac{1 + x}{1 + x + y}.$$

These irreducible forms cannot all together coincide with g/h. In general we can only say that

$$(Ph - gQ)(x, y) = \sum_{(i,j) \in N \bullet D \setminus I} e'_{ij} B_{ij}(x, y).$$

4. General Order Multivariate Newton–Padé Type Approximants

Newton–Padé type approximants are a special case of partial Newton–Padé approximants: the denominator polynomial is completely fixed and no factor of the numerator polynomial is prechosen. In the univariate case this means putting $\partial w_l = m$ and $\partial v_k = 0$ in (3), implying that $\partial p = n$ and $\partial q = 0$. The Newton–Padé type approximant is then given by

$$\frac{p}{w_m}(x) = \frac{\sum_{j=0}^{m} w_j \sum_{i=0}^{n} f[x_j, \ldots, x_i]\, B_i(x)}{\sum_{j=0}^{m} w_j B_j(x)}$$

satisfying

$$(f w_m - p)(x) = \sum_{i=n+1}^{\infty} d_i B_i(x).$$

Convergence results for Newton–Padé approximants can be found in [9, 10]. From (4) we find that its multivariate analogon is

$$\frac{P}{W_m}(x, y) = \frac{\sum_{(k,l)\in W} w_{kl} \sum_{(i,j)\in N} c_{kl,ij} B_{ij}(x, y)}{\sum_{(k,l)\in W} w_{kl} B_{kl}(x, y)}$$

with $c_{kl,ij} = f[x_k, ..., x_i][y_l, ..., y_j]$ and satisfying

$$(fW_m - P)(x, y) = \sum_{(i,j)\in \mathbb{N}^2 \setminus N} d_{ij} B_{ij}(x, y).$$

For general order multivariate Padé type approximants this last formula reduces to

$$\frac{P}{W_m}(x, y) = \frac{\sum_{(k,l)\in W} w_{kl} \sum_{(i,j)\in N} c_{i-k,j-l} x^j y^j}{\sum_{(k,l)\in W} w_{kl} x^k y^l}$$

with $c_{i-k,j-l} = (\partial^{i-k+j-l} f / \partial x^{i-k} \partial y^{j-l})(0, 0)$. A convergence theorem for multivariate Padé-type approximants will be given in [6]. We shall now rediscover some independently developed notions of multivariate Newton–Padé type or multivariate Padé type approximants as special cases of our general order rational approximants.

In [2] Brezinski introduces multivariate Padé type approximants with a denominator polynomial of the form

$$W_m(x, y) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} w_{ij} x^i y^j$$

satisfying

$$(fW_m - P)(x, y) = \sum_{(i,j)\in \mathbb{N} * (\mathbb{N} + \{(0,m_2)\}) \cup (\mathbb{N} + \{(m_1,0)\}) * \mathbb{N}} d_{ij} x^i y^j$$

which is equivalent to (4) with $I = N = ([0, m_1 - 1] \times [0, m_2 - 1]) \cap \mathbb{N}^2$. Here

$$\# W = (m_1 + 1)(m_2 + 1).$$

In [12] Kida introduces multivariate Padé type approximants using multivariate homogeneous expressions. Their construction is similar to the construction of multivariate Padé approximants by Cuyt in [5]. He chooses

$$W_m(x, y) = \sum_{i+j=s}^{s+m} w_{ij} x^i y^j$$

$$P(x, y) = \sum_{i+j=s}^{s+n} a_{ij} x^i y^j$$

with the multivariate Padé type approximant satisfying

$$(fW_m - P)(x, y) = \sum_{i+j=s+n+1}^{\infty} d_{ij} x^i y^j.$$

So $I = \{(i, j) \mid 0 \leqslant i+j \leqslant s+n\}$. The integer s indicates a shift of the degrees of W_m and P over s. Instead of the index sets W and N being triangular, they have a band structure, resulting from shifting the triangle away from the origin by s.

In [15, 16] Sablonniere chooses m points $z^{(1)} = (z_1^{(1)}, z_2^{(1)}), \ldots, z^{(m)} = (z_1^{(m)}, z_2^{(m)})$ in $\mathbb{C}^2$ to construct

$$W_m(x, y) = \prod_{i=1}^{m} (1 - z_1^{(i)} x - z_2^{(i)} y)$$

$$= \sum_{i+j=0}^{m} w_{ij} x^i y^j$$

while the numerator of the multivariate Padé type approximant is of the form

$$P(x, y) = \sum_{i+j=0}^{m-1} a_{ij} x^i y^j.$$

Hence

$$\# W = (m+1)(m+2)/2$$

$$\# N = m(m+1)/2.$$

Finally the approximation conditions are given by

$$(fW_m - P)(x, y) = \sum_{i+j=m}^{\infty} d_{ij} x^i y^j.$$

Recently Mühlbach introduced a multivariate Newton–Padé type approximant which he called in [14] a multivariate rational interpolant with prescribed poles. He fixes $m_1 + 1$ finite noncoinciding points $\alpha_{i,1}$ and $m_2 + 1$ finite noncoinciding points $\alpha_{j,2}$ in $\mathbb{C}$ to construct

$$W_m(x, y) = \prod_{i=0}^{m_1} (x - \alpha_{i,1}) \prod_{j=0}^{m_2} (y - \alpha_{j,2}).$$

The multivariate Newton–Padé type approximant under consideration will be computed in its partial fraction decomposition

$$\frac{P}{W_m}(x, y) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} \frac{\beta_{ij}}{(x - \alpha_{i,1})(y - \alpha_{j,2})}.$$

So

$$W = ([0, m_1] \times [0, m_2]) \cap \mathbb{N}^2$$
$$I = N = ([0, m_1 - 1] \times [0, m_2 - 1]) \cap \mathbb{N}^2.$$

The above results can be generalised for multiple prescribed poles, where some of the $\alpha_{i,1}$ or $\alpha_{j,2}$ coincide.

5. Numerical Illustration

To illustrate the concept of multivariate partial Padé approximant we shall compare it numerically with the general order multivariate Padé approximant. It is clear that the partial approximant can only be a powerful tool if the polynomials $W_l(x, y)$ and $V_k(x, y)$ contain accurate information on the zeros and poles of the multivariate function responsible for the interpolation data. The bivariate Beta function $B(x, y)$ will serve as a concrete example here because many numerical results on other types of approximants for this function can be found in the literature [4, 5, 7, 11, 13]. It is defined by

$$B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)},$$

where Γ is the Gamma function. Singularities occur at $x = -k$ and $y = -k$, $(k = 0, 1, 2, \ldots)$ and zeros at $y = -x - k$ $(k = 0, 1, 2, \ldots)$. By means of the recurrence formulas

$$\Gamma(x+1) = x\Gamma(x)$$
$$\Gamma(y+1) = y\Gamma(y)$$

for the Gamma function, we can write

$$B(x, y) = \frac{1 + (x-1)(y-1)\,f(x, y)}{xy}.$$

We shall now compute approximants $R(x, y)$ for $f(x, y)$ and compare the exact value $B(u_i, v_j)$ with the expression

$$\frac{1 + (u_i - 1)(v_j - 1)\,R(u_i, v_j)}{u_i v_j}$$

in a number of points (u_i, v_j) close to zeros and poles simulated by the

partial Padé approximant. For the partial approximant the supplementary information on the poles is given by

$$W_3(x, y) = (1 + x)(1 + y)$$
$$W = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$$

and that on the zeros by

$$V_2(x, y) = 1 + x + y$$
$$V = \{(0, 0), (1, 0), (0, 1)\}.$$

Of course, when comparing partial approximants with full approximants this extra information on the zeros and poles will be accounted for. For the general order Padé approximants in total 36 pieces of information shall be used, namely the Taylor coefficients c_{ij} of a series development for $f(x, y)$ with (i, j) in

$$I = \{(i, j) \mid 0 \leqslant i \leqslant 5, 0 \leqslant j \leqslant 5\}.$$

For the numerator and denominator of the general order Padé approximant we take

$$N = \{(i, j) \mid 0 \leqslant i + j \leqslant 5\} \cup \{(3, 3)\}$$
$$D = \{(i, j) \mid 0 \leqslant i + j \leqslant 4\}.$$

For the numerator and denominator of the partial Padé approximant we keep in mind that already some coefficients are fixed by the choice of W_3 and V_2 and hence we take

$$N = \{(i, j) \mid 0 \leqslant i + j \leqslant 5\} \cup \{(3, 3)\} \setminus \{(5, 0), (0, 5)\}$$
$$D = \{(i, j) \mid 0 \leqslant i + j \leqslant 3\} \cup \{(3, 1), (3, 1)\}.$$

The unknown coefficients in the partial Padé approximant can be fixed by imposing $36 - k - l = 36 - 2 - 3 = 31$ approximation conditions coming from

$$I = \{(i, j) \mid 0 \leqslant i \leqslant 4, 0 \leqslant j \leqslant 4\} \cup \{\{(5, 0), (5, 1), (5, 2), (2, 5), (1, 5), (0, 5)\}.$$

Note that our approximants are chosen symmetric in x and y because we are dealing with a symmetric function. Table I displays some numerical results that are typical throughout the region $[-1, 1] \times [-1, 1]$. We did not pick particular numbers that served our purpose. The difficulty with the bivariate Beta function is that it is very steep near its zeros and singularities. This forces us to go quite close to illustrate the advantage

TABLE I

(u_i, v_j)	$R(u_i, v_j) = [N/D]_l^l$ $E_{14}^{(21)}$	$R(u_i, v_j) = \{N, V/D, W\}_l^l$ $E_{11}^{(19)}$	$B(u_i, v_j)$
$(-0.9875, -0.2)$	56.	110.	91.
$(-0.9925, -0.2)$	75.4	196.	155.
$(-0.9975, -0.2)$	112.	603.	475.
$(-0.5975, -0.4)$	−0.0334	−0.0346	−0.0343
$(-0.6, -0.4)$	0.0009	0.	0.
$(-0.6025, -0.4)$	0.0355	0.0349	0.0345
$(-0.9800, -0.25)$	42.5	68.3	58.3
$(-0.9875, -0.25)$	59.0	113.	95.8
$(-0.9950, -0.25)$	93.4	290.	246.
$(-0.2, -0.7990)$	−0.041	−0.036	−0.033
$(-0.8, -0.1990)$	−0.049	−0.039	−0.034
$(-0.85, -0.149)$	−0.120	−0.076	−0.055

of the partial Newton–Padé approximants. The displayed approximants were computed using the *E*-algorithm for multivariate Newton–Padé approximants [4].

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